Supercurrent induced by tunneling Bogoliubov excitations in a Bose-Einstein condensate

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Abstract

We study the tunneling of Bogoliubov excitations through a barrier in a Bose-Einstein condensate. We extend our previous work [Phys. Rev. A **78**, 013628 (2008)] to the case when condensate densities are different between the left and right of the barrier potential. In the framework of the Bogoliubov mean-field theory, we calculate the transmission probability and phase shift, as well as the energy flux and quasiparticle current carried by Bogoliubov excitations. We find that Bogoliubov phonons twist the condensate phase due to a back-reaction effect, which induces the Josephson supercurrent. While the total current given by the sum of quasiparticle current and induced supercurrent is conserved, the quasiparticle current flowing through the barrier potential is shown to be remarkably enhanced in the low energy region. When the condensate densities are different between the left and right of the barrier, the excess quasiparticle current, as well as the induced supercurrent, remains finite far away from the barrier. We also consider the tunneling of excitations and atoms through the boundary between the normal and superfluid regions. We show that supercurrent can be generated inside the condensate by injecting free atoms from outside. On the other hand, atoms are emitted when the Bogoliubov phonons propagate toward the phase boundary from the superfluid region.

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I. INTRODUCTION

In the pioneering work by Bogoliubov [1], it was shown that the Bose-Einstein condensate (BEC) of weakly interacting bosons has a phonon-type excitation mode. It is now called the Bogoliubov mode, which is a Nambu-Goldstone mode associated with a spontaneous broken U(1) symmetry [2]. This collective mode dominates low-energy properties of BEC, so that it is an important key to understand physical properties of BEC [3]. In particular, the existence of Bogoliubov phonon is essential for the Bose-condensed phase to acquire superfluidity [4]. Since the realization of BECs in ultracold atomic gases [5, 6], the study of Bogoliubov mode has been one of the main issues in cold atom physics [3, 7]. Because of the high degree of controllability, the BECs of cold atomic gases offer good opportunities to explore novel properties of Bogoliubov excitations.

Recently, Kovrizhin and co-workers [8, 9, 10] predicted that the Bogoliubov mode exhibits striking tunneling properties. They showed that the transmission probability of Bogoliubov phonon through a potential barrier increases in the low energy region with decreasing the incident energy. In the low-energy limit, the perfect transmission is realized irrespective of the height of the barrier. This interesting tunneling property of Bogoliubov mode is referred to as the *anomalous tunneling* [10]. Since their prediction [8, 9, 10], the anomalous tunneling has attracted much attention, and has been addressed by many papers [11, 12, 13, 14, 15, 16, 17, 18, 19].

As the origin of the anomalous tunneling effect, various mechanisms have been proposed, such as quasiresonance scattering [10], localized components of Bogoliubov mode appearing near the barrier [11], and anomalous enhancement of quasiparticle current [12]. For the perfect transmission in the low-energy limit, the importance of the coincidence of the condensate and excitation wave functions [13], as well as supercurrent behavior of low-energy Bogoliubov phonons [14], has been pointed out. The anomalous tunneling phenomenon was shown to occur even in the supercurrent state [11, 14, 20], as well as at finite temperatures [13]. It has been also studied in the presence of a periodic potential [15, 16], as well as a random potential [17]. It has been also pointed out that similar phenomena to this can be seen in the scattering of Bogoliubov phonon by a spherical potential in three dimensions d[18], as well as the refraction of Bogoliubov phonons [19].

In this paper, we investigate tunneling properties of Bogoliubov phonon in a BEC at

$T = 0$. In Ref. [12], we have considered the case when the incident and transmitted Bogoliubov phonons feel the same condensate densities on both the right and left of the barrier. In this paper, we extend this previous paper to the case when the condensate density is different between the right and the left of the barrier. As an extreme case, we also deal with the case when the condensate density is absent on one side of the barrier. Applying the finite element method to the Bogoliubov coupled equations, we numerically calculate the transmission probability and phase shift of Bogoliubov phonons. We find that Bogoliubov phonons twist the phase of the BEC order parameter (condensate wave function) due to a back-reaction effect, which leads to the induction of Josephson supercurrent. The induced supercurrent is shown to satisfy the Josephson relation with respect to the twisted phase when the condensate density is the same on both sides of the barrier. The supercurrent is induced only in the region near the barrier when the condensate has the same densities across the potential barrier. In the case when the condensate density is different between the right and left of the barrier, the supercurrent is also induced in the region far away from the barrier. In addition, the excess quasiparticle current is supplied from the condensate to conserve the total current, so that one obtains the enhancement of the transmission probability of quasiparticle current in the low-energy region. We also show that the supercurrent is induced when one injects free atoms from the outside of condensate. In addition, atoms are shown to evaporate from the surface of superfluid region when Bogoliubov excitation propagates toward the superfluid-normal phase boundary.

This paper is organized as follows: in Sec. II, we present the model and formalism of the Bogoliubov mean-field approximation, as well as the finite element method which we apply for solving the Bogoliubov equations. In Sec. III, we study the tunneling of Bogoliubov phonons through a rectangular potential barrier. We give a detailed discussion on the origin of the anomalous tunneling and induced Josephson supercurrent. In Sec. IV, we study the tunneling in the presence of a step potential which yields the different condensate densities across the potential barrier. In Sec. V, we discuss the tunneling of excitations and atoms between superfluid and normal regions.

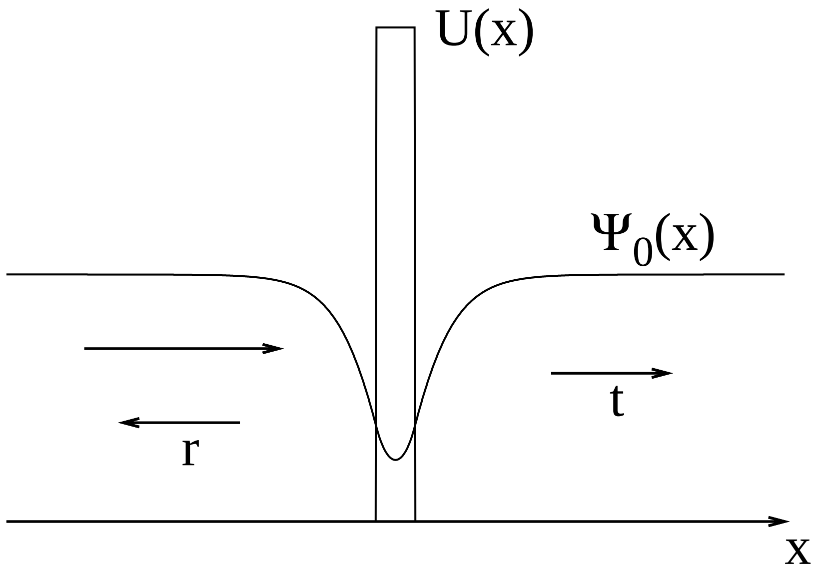

FIG. 1: Schematic of the system. The arrows on the left describe the incoming (upper arrow) and reflected (lower arrow) Bogoliubov excitations. The arrow on the right describes the transmitted one.

II. MODEL AND FORMALISM

We consider tunneling phenomena of Bogoliubov mode through a barrier potential, as schematically shown in Fig. 1. We assume that the barrier potential $U(x)$ only depends on x and ignore the motion of atoms in the y and z directions, so that we consider a one-dimensional tunneling problem along the x direction. This kind of one-dimensional geometry has been recently realized [21]. In Ref. [21], a BEC was prepared in a narrow elongated trap with a wall-type potential barrier, which varies only in the axial direction and the potential width is much longer than the radial size of the gas cloud. We also ignore temperature effects as well as effects of a harmonic trap. The latter assumption is justified when the BEC is trapped in an elongated trap [21, 22] or a box-shaped trap [23].

We treat the tunneling of Bogoliubov mode within the Bogoliubov mean-field theory for a weakly interacting Bose gas at $T = 0$ [1, 24, 25]. To describe the BEC phase, we divide the Bose field operator $\hat{\psi}(\boldsymbol{r})$ into the condensate wave function $\Psi_0(\boldsymbol{r})$ and the noncondensate

part, as

$$\hat{\psi}(\boldsymbol{r}) = \Psi_0(\boldsymbol{r}) + \sum_j \left[u_j(\boldsymbol{r})\hat{\alpha}_j - v_j(\boldsymbol{r})^* \hat{\alpha}_j^\dagger \right], \tag{1}$$

where $\hat{\alpha}_j^\dagger$ is the creation operator of a Bogoliubov excitation in the jth state. The condensate wave function $\Psi_0(\boldsymbol{r}) = \langle \hat{\psi}(\boldsymbol{r}) \rangle$ satisfies the static Gross-Pitaevskii (GP) equation [24, 26],

$$\left(-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + g|\Psi_0|^2 \right) \Psi_0 = \mu \Psi_0. \tag{2}$$

Here, m, μ, and $U(\boldsymbol{r})$ represent the mass of a boson, chemical potential, and barrier potential, respectively. $g(> 0)$ is a repulsive interaction between bosons. In Eq. (1), $u_j(\boldsymbol{r})$ and $v_j(\boldsymbol{r})$ satisfy the Bogoliubov coupled equations,

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + 2g|\Psi_0|^2 - \mu \right] u_j - g\Psi_0^2 v_j = E_j u_j, \tag{3}$$

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + 2g|\Psi_0|^2 - \mu \right] v_j - g(\Psi_0^*)^2 u_j = -E_j v_j, \tag{4}$$

where E_j is the Bogoliubov excitation spectrum. To solve Eqs. (3) and (4) with an appropriate boundary condition, we use the finite element method [27]. For this purpose, it is convenient to rewrite Eqs. (3) and (4) in the matrix form

$$\left(\hat{\bar{H}} - \bar{E}_j \tau_3 \right) \phi_j = 0, \tag{5}$$

where

$$\hat{\bar{H}} = \begin{pmatrix} -\bar{\nabla}^2 + \bar{U}(\bar{\boldsymbol{r}}) + 2|\bar{\Psi}_0|^2 - 1 & -\bar{\Psi}_0^2 \\ -(\bar{\Psi}_0^*)^2 & -\bar{\nabla}^2 + \bar{U}(\bar{\boldsymbol{r}}) + 2|\bar{\Psi}_0|^2 - 1 \end{pmatrix}, \tag{6}$$

$$\phi_j = \begin{pmatrix} u_j(\bar{\boldsymbol{r}}) \\ v_j(\bar{\boldsymbol{r}}) \end{pmatrix}, \tag{7}$$

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

In Eq. (5), we have introduced dimensionless variables $\bar{\boldsymbol{r}} \equiv \boldsymbol{r}/\xi$, $\bar{E}_j \equiv E_j/\mu$, $\bar{U} \equiv U/\mu$, and $\bar{\Psi}_0 \equiv \Psi_0/n_0$, where $n_0 \equiv \mu/g$ is the condensate density far away from the barrier, and $\xi \equiv 1/\sqrt{2mgn_0}$ is the healing length. To simplify our notations, we omit the bars and indices of eigenstates in the following part of this section. Equation (5) can be obtained from the variational principle $\delta L = 0$, when the Lagrangian L has the form

$$L = \int_\Omega d\boldsymbol{r} \left[(\nabla \phi^\dagger) \cdot (\nabla \phi) + \phi^\dagger \left(U(\boldsymbol{r}) + 2|\Psi_0|^2 - 1 - \Psi_0^2 \tau_+ - (\Psi_0^*)^2 \tau_- \right) \phi - E\tau_3 \right]. \tag{9}$$

Here, Ω is the volume of the system, and $\tau_{\pm}$ are given by

$$\tau_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

We introduce N spatial positions $\boldsymbol{r}_i$ $(i = 1, 2, \ldots, N)$ in the system, which are referred to as nodes in the literature of the finite element method [27]. We then assign the interpolation function $N_i(\boldsymbol{r})$ at each $\boldsymbol{r}_i$, which equals unity at $\boldsymbol{r} = \boldsymbol{r}_i$ and linearly decreases to zero at adjacent nodes of $\boldsymbol{r}_i$. Namely, the interpolation function satisfies

$$N_i(\boldsymbol{r}_j) = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases} \tag{11}$$

For example, in one-dimensional case, we define the nodes at x_i $(i = 1, 2, \ldots, N)$. The interpolation function $N_i(x)$ is given by

$$N_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & (x_{i-1} \leq x \leq x_i), \\ -\frac{x - x_{i+1}}{x_{i+1} - x_i}, & (x_i \leq x \leq x_{i+1}), \\ 0, & (x < x_i, x > x_{i+1}). \end{cases} \tag{12}$$

Using $N_i(\boldsymbol{r})$, one can approximately write $u(\boldsymbol{r})$ and $v(\boldsymbol{r})$ in the forms

$$u(\boldsymbol{r}) = \sum_i u_i N_i(\boldsymbol{r}), \tag{13}$$

$$v(\boldsymbol{r}) = \sum_i v_i N_i(\boldsymbol{r}). \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (9), we obtain

$$L = \sum_{i,j} \left[u_i^* \left(K_{i,j} - M_{i,j}^{-} \right) u_j + v_i^* \left(K_{i,j} + M_{i,j}^{+} \right) v_j - u_i^* P_{i,j} v_j - v_i^* P_{i,j}^* u_j \right], \tag{15}$$

where

$$
\begin{aligned}
K_{i,j} &= \int_{\Omega_{i,j}} d\boldsymbol{r}\ \left(\nabla N_i\right)\cdot\left(\nabla N_j\right), &(16)\\
M^{\pm}_{i,j} &= \int_{\Omega_{i,j}} d\boldsymbol{r}\ N_i\left[E \pm \left(U(\boldsymbol{r}) + 2|\Psi_0|^2 - 1\right)\right] N_j \\
&= (E \mp 1)\int_{\Omega_{i,j}} d\boldsymbol{r}\ N_i N_j \pm \sum_l U_l \int_{\Omega_{i,j,l}} d\boldsymbol{r}\ N_i N_j N_l \\
&\quad \pm 2\sum_{l,l'} (f_l - i g_l)(f_{l'} + i g_{l'}) \int_{\Omega_{i,j,l,l'}} d\boldsymbol{r}\ N_i N_j N_l N_{l'}, &(17)\\
P_{i,j} &= \int_{\Omega_{i,j}} d\boldsymbol{r}\ N_i \Psi_0^2 N_j \\
&= \sum_{l,l'} (f_l + i g_l)(f_{l'} + i g_{l'}) \int_{\Omega_{i,j,l,l'}} d\boldsymbol{r}\ N_i N_j N_l N_{l'}. &(18)
\end{aligned}
$$

In obtaining Eqs. (17) and (18), we have expanded $U(\boldsymbol{r})$ and $\Psi_0(\boldsymbol{r})$ as

$$
\begin{aligned}
U(\boldsymbol{r}) &= \sum_l U_l N_l(\boldsymbol{r}), &(19)\\
\Psi_0(\boldsymbol{r}) &= \sum_l (f_l + i g_l) N_l(\boldsymbol{r}). &(20)
\end{aligned}
$$

Here, $U_l = U(\boldsymbol{r}_l)$, $f_l = \mathrm{Re}\left[\Psi_0(\boldsymbol{r}_l)\right]$, and $g_l = \mathrm{Im}\left[\Psi_0(\boldsymbol{r}_l)\right]$. In Eqs. (16)-(18), $\Omega_{i,j}$, $\Omega_{i,j,l}$, and $\Omega_{i,j,l,l'}$ mean that the integrations are carried out in the regions where $N_i N_j$, $N_i N_j N_l$, and $N_i N_j N_l N_{l'}$ are finite, respectively. The integrations in Eqs. (16)-(18) can be evaluated in the standard manner of the finite element method [27].

Equation (15) can be rewritten in the matrix form as

$$
L = \boldsymbol{u}^\dagger \left(\hat{K} - \hat{M}^-\right) \boldsymbol{u} + \boldsymbol{v}^\dagger \left(\hat{K} + \hat{M}^+\right) \boldsymbol{v} - \boldsymbol{u}\hat{P}\boldsymbol{v} - \boldsymbol{v}^\dagger \hat{P}^* \boldsymbol{u}, \tag{21}
$$

where $\{\boldsymbol{u}\}_i = u_i$, $\{\boldsymbol{v}\}_i = v_i$, $\{\hat{K}\}_{i,j} = K_{i,j}$, $\{\hat{M}^\pm\}_{i,j} = M^\pm_{i,j}$, and $\{\hat{P}\}_{i,j} = P_{i,j}$. The equations for $\boldsymbol{u}$ and $\boldsymbol{v}$ are, respectively, obtained from $\delta L/\delta \boldsymbol{u}^\dagger = 0$ and $\delta L/\delta \boldsymbol{v}^\dagger = 0$, which give

$$
\begin{aligned}
\left(\hat{K} - E\hat{M}^-\right)\boldsymbol{u} - \hat{P}\boldsymbol{v} &= 0, &(22)\\
\left(\hat{K} + E\hat{M}^+\right)\boldsymbol{v} - \hat{P}\boldsymbol{u} &= 0. &(23)
\end{aligned}
$$

The advantage of using the finite element method is that one can obtain the solutions by simply diagonalizing Eqs. (22) and (23) under an appropriate boundary condition, instead of solving the differential Eqs. (3) and (4). In the following sections, we will numerically solve Eqs. (22) and (23) for given barrier potentials.

III. TUNNELING THROUGH THE RECTANGULAR POTENTIAL BARRIER

In this section, we consider the one-dimensional tunneling problem of Bogoliubov excitations through a rectangular barrier potential shown in Fig. 1. The potential barrier is given by

$$U(x) = U_0\ \theta\left(\frac{d}{2} - |x|\right), \tag{24}$$

where $\theta(x)$ is the step function. U_0 and d describe the height and width of the barrier, respectively, and we consider the case of repulsive potential barrier ($U_0 > 0$). In this section, we treat the case when the condensate densities are the same on both sides of the barrier, as shown in Fig. 1. Although this case has been examined in our previous paper [12], we give further analyses for the tunneling of Bogoliubov phonon here. In Secs. IV and V, we will also compare the results in this section with the case when the condensate density on the left of the barrier is different from that on the right of the barrier.

In the present case, the GP equation can be solved analytically [10], as

$$\bar{\Psi}_0(\bar{x}) = \begin{cases} \tanh\left[\frac{1}{\sqrt{2}}\left(|\bar{x}| - \frac{\bar{d}}{2}\right) + \mathrm{arctanh}\gamma\right], & (|x| \geq d/2), \\ \frac{\beta}{\mathrm{cn}\left(\sqrt{\frac{K^2+\beta^2}{2}}\bar{x}, q\right)}, & (|x| < d/2). \end{cases} \tag{25}$$

Here, $\bar{d} \equiv d/\xi$, $K \equiv \sqrt{\beta^2 + 2(\bar{U}_0 - 1)}$, and $q \equiv K/\sqrt{K^2 + \beta^2}$, where $\bar{U}_0 \equiv U_0/\mu$. ($\mathrm{cn}(x, q)$ is the Jacobi's elliptic function [28].) $\gamma \equiv \bar{\Psi}_0(\bar{x} = \bar{d}/2)$ and $\beta \equiv \bar{\Psi}_0(0)$ are determined from the boundary conditions in terms of $\Psi_0(x)$ and $d\Psi_0/dx$ at $x = \pm d/2$, which give

$$\gamma = \frac{\beta}{\mathrm{cn}\left(\sqrt{\frac{K^2+\beta^2}{2}}\frac{\bar{d}}{2}, q\right)}, \tag{26}$$

$$\gamma^2 = \frac{1}{2\bar{U}_0}\left(\beta^4 + 2(\bar{U}_0 - 1)\beta^2 + 1\right). \tag{27}$$

The values β and γ are determined by numerically solving Eqs. (26) and (27).

To solve Bogoliubov equations (22) and (23), we need asymptotic solutions for $x = \pm\infty$. In our tunneling problem, each eigenstate with index j in Eqs. (3) and (4) corresponds to Bogoliubov excitation with energy E injected from one side of the barrier. In Secs. III-V, we omit the index for eigenstates for simplicity. Far from the barrier ($|x| \gg \xi$), the Bogoliubov mode is described by the plane-wave $(u(x), v(x)) = (u_E, v_E)e^{ipx}$. Substituting this into Eqs. (3) and (4), one obtains the well-known Bogoliubov excitation spectrum as [1]

$$E_p = \sqrt{\varepsilon_p(\varepsilon_p + 2gn_0)}, \tag{28}$$

where $\varepsilon_p = p^2/2m$. Namely, for a given mode energy E, there are four particular solutions in terms of the momentum p, given by

$$p = \begin{cases} \pm\sqrt{2m}\sqrt{\sqrt{E^2+(gn_0)^2} - gn_0} \equiv \pm k, \\ \pm i\sqrt{2m}\sqrt{\sqrt{E^2+(gn_0)^2} + gn_0} \equiv \pm i\kappa. \end{cases} \tag{29}$$

The first two solutions ($p = \pm k$) describe the ordinary propagating waves in the $\pm x$-directions. The remaining two imaginary solutions ($p = \pm i\kappa$) describe localized states. We note that while the latter localized solutions are actually not necessary in a homogeneous system, we cannot ignore them in the present inhomogeneous system. The amplitudes of the propagating components are given by

$$\begin{pmatrix} u_E^{\mathrm{P}} \\ v_E^{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2L}\left(\frac{\sqrt{E^2+(gn_0)^2}}{E}+1\right)} \\ \sqrt{\frac{1}{2L}\left(\frac{\sqrt{E^2+(gn_0)^2}}{E}-1\right)} \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix}, \tag{30}$$

where L is the system size in the x direction. On the other hand, the amplitudes for the localized states are given by $(u_E^{\mathrm{L}}, v_E^{\mathrm{L}}) = (-b, a)$. Thus, in contrast to the propagating solution in Eq. (30), the normalization of the localized components becomes negative as $(u_E^{\mathrm{L}})^2 - (v_E^{\mathrm{L}})^2 = -1/L$.

Using the propagating solution $(u_E^{\mathrm{P}}, v_E^{\mathrm{P}})e^{\pm ikx}$ and localized one $(u_E^{\mathrm{L}}, v_E^{\mathrm{L}})e^{\pm\kappa x}$, we construct the asymptotic forms of the Bogoliubov wave function for $x \to \pm\infty$. Assuming that the Bogoliubov phonon is injected from $x = -\infty$, we obtain the asymptotic solutions as

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{ikx} + r\begin{pmatrix} a \\ b \end{pmatrix} e^{-ikx} + A\begin{pmatrix} -b \\ a \end{pmatrix} e^{\kappa x}, \quad (x \to -\infty)\ , \\ \begin{pmatrix} u \\ v \end{pmatrix} = t\begin{pmatrix} a \\ b \end{pmatrix} e^{ikx} + B\begin{pmatrix} -b \\ a \end{pmatrix} e^{-\kappa x}, \quad (x \to \infty)\ . \end{cases} \tag{31}$$

Here, r and t are, respectively, the reflection and transmission amplitudes, which satisfy

$$|r|^2 + |t|^2 = 1. \tag{32}$$

As will be discussed later, this condition is deeply related to the conservation of energy flux. In Eq. (31), A and B represent the amplitudes of the localized components near the potential barrier.

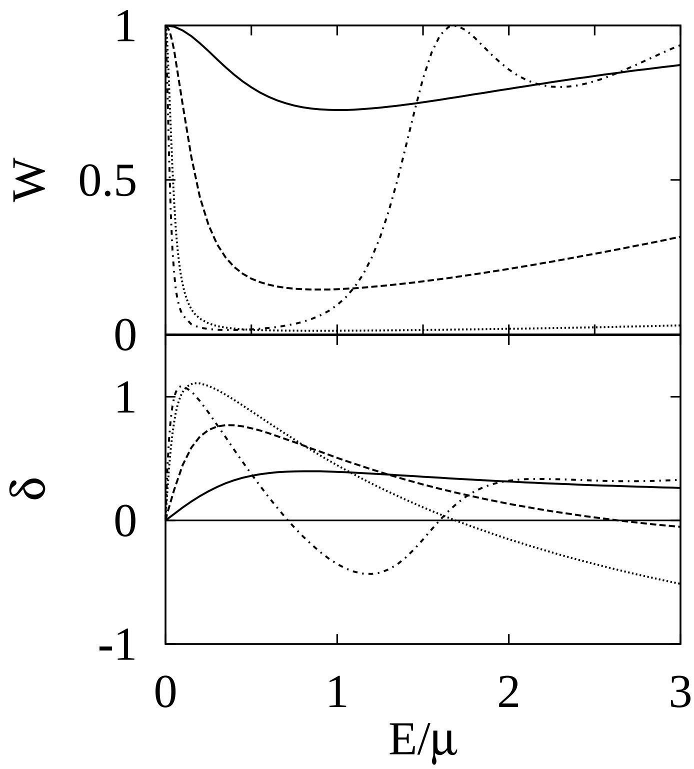

FIG. 2: Calculated transmission probability W and phase shift δ as functions of the incident energy E for a rectangular potential barrier. We set the width d and height U_0 of the barrier as $(d, U_0) = (\xi, 2\mu)$ (solid line), $(\xi, 5\mu)$ (dashed line), $(\xi, 10\mu)$ (dotted line), and $(4\xi, 2\mu)$ (dash-dotted line).

We numerically solve the Bogoliubov coupled Eqs. (22) and (23) for a given incident energy E. In this procedure, the condensate wave function in Eq. (25) is used, and the solution is determined so as to satisfy the asymptotic solution in Eq. (31).

Figure 2 shows the calculated transmission probability $W \equiv |t|^2$, as well as phase shift $\delta \equiv \arg(t)$, as functions of the incident energy E for various barrier heights and widths. We call attention to the characteristic features of W and δ in the low-energy region ($E/\mu \lesssim 0.5$). One can clearly see the anomalous tunneling behavior discussed in [8, 9, 10] in Fig. 2. Namely, below a certain incident energy ($E/\mu \sim 0.5$), W increases and δ decreases with decreasing E, in contrast to the behaviors above that energy (W decreases and δ increases as E decreases). Furthermore, W and δ approach unity and zero in the low-energy limit $E \to 0$, respectively, irrespective of the values of d and U_0. When the incident energy E is very large ($E \gg \mu$), since the Bogoliubov phonon loses its collective nature, the tunneling

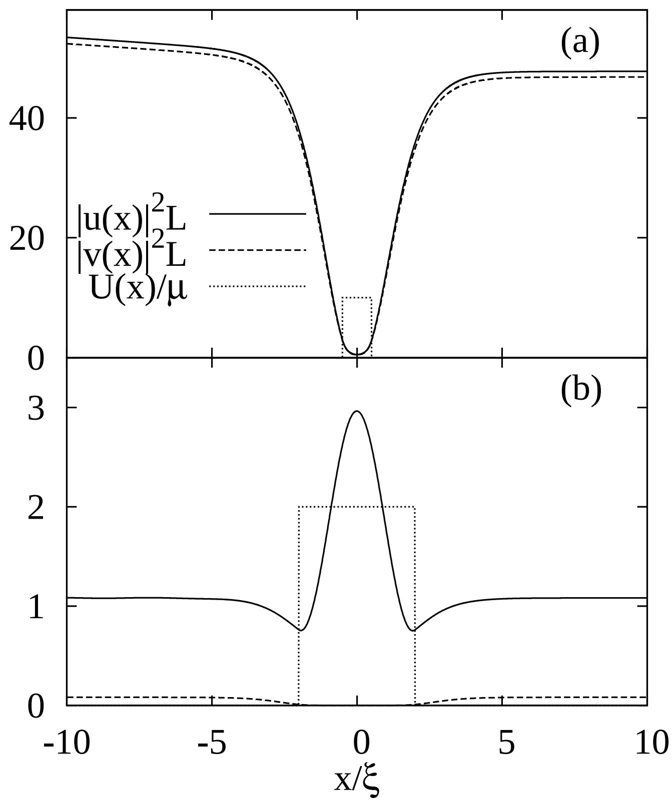

FIG. 3: Spatial variation in the Bogoliubov wave function $(u(x), v(x))$. (a) $E/\mu = 0.01 \ll 1$ (anomalous tunneling). We set $(d, U_0) = (\xi, 10\mu)$. (b) $E/\mu = 1.68$. We set $(d, U_0) = (4\xi, 2\mu)$. In this case, the resonance tunneling ($W = 1$) is realized, as shown in Fig. 2 (see the dash-dotted line). The dotted line is the potential barrier $U(x)$ in units of μ.

property becomes close to that of a single particle.

We note that the *perfect transmission* of Bogoliubov phonon ($W \to 1, \delta \to 0$) shown in the low-energy limit in Fig. 2 is quite different from the typical tunneling properties of a single particle, where W and δ approach 0 and $-\pi/2$ in the low-energy limit, respectively. Namely, in the latter case the particle is completely reflected by the potential barrier [29].

We also note that the energy region in which W and δ exhibit the anomalous tunneling behavior (W increases and δ decreases with decreasing E) depends on the height U_0 and width d of the potential barrier. This region becomes narrower for higher and wider potential barrier, as shown in Fig. 2.

In Fig. 2, we find that $W = 1$ is also obtained at finite energy ($E/\mu = 1.68$) in the case of $(d, U_0) = (4\xi, 2\mu)$, due to the resonance tunneling effect. To see the difference between the resonance tunneling effect and the anomalous tunneling effect, we show in

Fig. 3 the wave functions in the two cases. In the case of resonance tunneling, one sees that while $|u|^2$ is enhanced in the barrier, $|v|^2$ is suppressed there. The peak of $|u|^2$ is a clear signature of the formation of a resonance state. The suppression of $|v|^2$ indicates that the Bogoliubov excitation behaves like a single particle during the tunneling through the barrier. In contrast, in the case of the anomalous tunneling, both $|u|^2$ and $|v|^2$ simply become small in the barrier and almost coincide with each other. Indeed, it was shown in Refs. [13, 14] that $u(x)$ and $v(x)$ reduce to the condensate wave function $\Psi_0(x)$ in the low-energy limit. The difference mentioned above indicates that the anomalous tunneling and resonance tunneling are different phenomena.

We briefly note that, as shown in Ref. [14], the anomalous tunneling effect originates from the fact that the wave functions of a Bogoliubov phonon with a small momentum p has the same form as the condensate wave function in the *supercurrent state*, accompanied by a finite superflow $J_s = n_0 p/m$. Recently, Morgan *et al.* [30] have presented a modified Bogoliubov theory where the wave function of Bogoliubov mode is constructed so as to be orthogonal to the solution obtained from the GP equation. Since their formalism does not affect the current-carrying component of the Bogoliubov wave function (which dominates the anomalous tunneling phenomenon), the perfect transmission of low-energy Bogoliubov phonon is still expected to occur. Thus, the anomalous tunneling phenomenon does not depend on the definition of the wave function of Bogoliubov mode.

Propagation of Bogoliubov phonon is accompanied by quasiparticle current $J_{\rm q}$, as well as energy flux $Q_{\rm q}$. When one uses the asymptotic solutions in Eq. (31), they are given by

$$J_{\rm q} = \begin{cases} \frac{k}{mL}(1-|r|^2) - \frac{2ab}{mL}e^{\kappa x}\left(\kappa \ {\rm Im}\left[A(e^{-ikx}+r^* e^{ikx})\right] + k \ {\rm Re}\left[A(e^{-ikx}-r^* e^{ikx})\right]\right), \\ \qquad\qquad (x \ll -\xi), \\ \frac{k}{mL}|t|^2 - \frac{2ab}{mL}e^{-\kappa x}\left(\kappa \ {\rm Im}\left[tB^* e^{ikx}\right] + k \ {\rm Re}\left[tB^* e^{ikx}\right]\right), \qquad (x \gg \xi), \end{cases} \tag{33}$$

$$Q_{\rm q} = \begin{cases} \frac{kE}{m}(a^2+b^2)(1-|r|^2), & (x \ll -\xi), \\ \frac{kE}{m}(a^2+b^2)|t|^2, & (x \gg \xi). \end{cases} \tag{34}$$

The detailed definitions of $J_{\rm q}$ and $Q_{\rm q}$ are summarized in Appendix B. Since the energy flux $Q_{\rm q}$ is conserved (see Appendix B), one obtains $|r|^2+|t|^2 = 1$ from Eq. (34). Using this, we find that the quasiparticle current $J_{\rm q}$ is also conserved in both limits $x = \pm\infty$ as $J_{\rm q}(x=-\infty) = k(1-|r|^2)/mL = J_{\rm q}(x=\infty) = k|t|^2/mL$. However, except for the limits $x = \pm\infty$, the last terms in Eq. (33) become finite, which come from the coupling between

the propagating and localized components in Eq. (31). As a result, while $Q_{\rm q}$ is conserved everywhere, we expect that $J_{\rm q}$ is not conserved near the barrier.

To see the non-conserving behavior of $J_{\rm q}$, we directly evaluate it using the solution of Bogoliubov equations (22) and (23). As shown in Fig. 4(a), we obtain the *excess* quasiparticle current

$$\Delta J_{\rm q}(x) \equiv J_{\rm q}(x) - J_{\rm q}(x=-\infty) \tag{35}$$

near the barrier. Namely, when the Bogoliubov phonon approaches the barrier, $J_{\rm q}$ is enhanced. $J_{\rm q}$ is constant in the barrier, and it decreases to be $J_{\rm q}(\infty) = J_{\rm q}(-\infty)$ when the phonon goes away from the barrier. In Fig. 4(a), the enhancement of $\Delta J_{\rm q}$ occurs near the barrier where the condensate density $n_{\rm s}(x)$ deviates from $n_0[= n_{\rm s}(x=\pm\infty)]$.

In Fig. 4(a), we find that the enhancement of $\Delta J_{\rm q}$ is more pronounced for lower incident energy E. In addition, as shown in Fig. 5, the excess quasiparticle current is more remarkable when the barrier is high, although the energy region where $\Delta J_{\rm q}(x=0)$ is large is narrower for larger U_0. Since $\Delta J_{\rm q}(x=0)/(k/mL)$ approaches a constant value, we also find from Fig. 5 that $\Delta J_{\rm q}(x=0)$ is proportional to the incident momentum k in the low-energy limit. [Note that $\Delta J_{\rm q}(x=0)$ in Fig. 5 is normalized by the incident quasiparticle current k/mL.]

The enhancement of quasiparticle current near the potential barrier implies that more quasiparticles than those carried in the incident current impinge on the barrier. Apparently, this is expected to lead to the increase in the transmission probability of quasiparticles. Indeed, comparing the result for $(d, U_0) = (\xi, 10\mu)$ in Fig. 4 with the corresponding result in Fig. 2, one finds that the energy region where the anomalous enhancement of transmission probability is obtained ($E \lesssim 0.1\mu$) coincides with the region where the excess quasiparticle current $\Delta J_{\rm q}(x=0)$ is remarkable.

As shown in Ref. [12], the excess quasiparticle current is supplied from the condensate. Namely, the transmission of Bogoliubov phonon is considered to be assisted by the supply of excess current from the condensate. Thus, in a sense, the mechanism of the anomalous tunneling may be considered as a kind of screening effect by Bose condensate. This argument partially explains the physical mechanism of the anomalous tunneling effect discussed in Refs. [8, 9, 10]. However, apart from the enhancement of low-energy transmission probability, this argument is not enough to explain the *perfect* transmission in the low-energy limit. In this regard, in Ref. [14], we have shown that the perfect transmission can be understood as a result of the supercurrent behavior of low-energy Bogoliubov phonon.

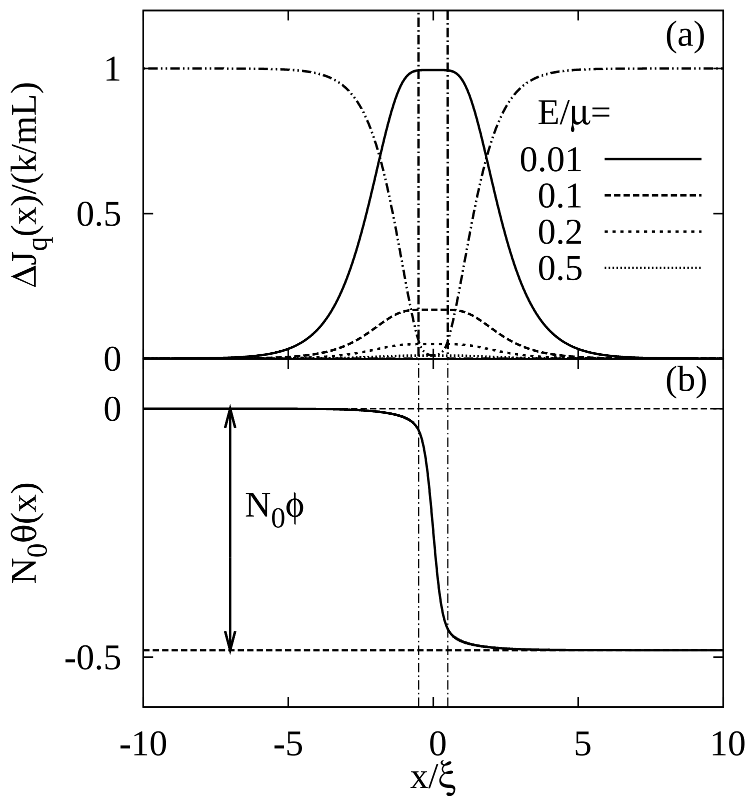

FIG. 4: (a): Excess quasiparticle current $\Delta J_{\rm q}(x) \equiv J_{\rm q}(x) - J_{\rm q}(x=-\infty)$ when $(d, U_0) = (\xi, 10\mu)$. The dash-double dotted line is the condensate density $n_{\rm s}(x)$ in units of n_0. (b): Phase $\theta(x)$ of the condensate wave function $\Psi_0(x)$ created by the tunneling of Bogoliubov phonon. We set $\theta(x=-\infty) = 0$. ϕ is the phase difference between condensates at $x = \pm\infty$. $N_0 = n_0 L$ is the number of condensate atoms. In both the panels, the dash-dotted line indicates the region of the potential barrier.

In Ref. [12], it was found that the counterflow of supercurrent is induced near the potential barrier due to a back-reaction effect of quasiparticle current, which restores the conservation of total current. The induction of supercurrent indicates that the phase of the BEC order parameter $\Psi_0(x)$ is twisted by quasiparticle current as

$$\Psi_0(x) \to e^{i\theta(x)}\Psi_0(x). \tag{36}$$

(Here, we assume that the amplitude of the condensate wave function is unchanged.) The induced supercurrent by this phase modulation is given by

$$\Delta J_{\rm s}(x) = \frac{n_{\rm s}(x)}{m}\partial_x \theta(x). \tag{37}$$

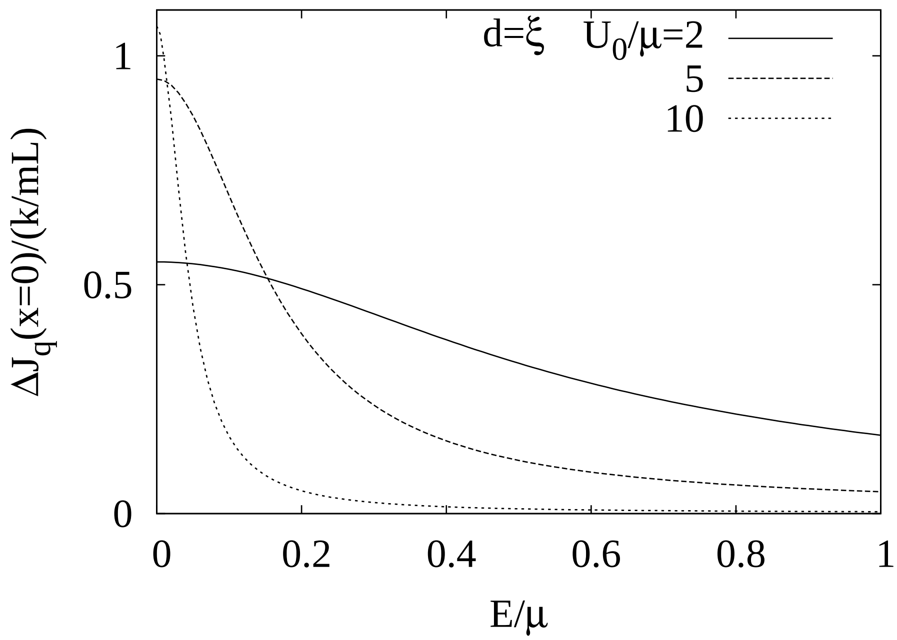

FIG. 5: Excess quasiparticle current in the potential barrier $\Delta J_{\rm q}(x=0)$ as a function of the incident energy E.

As shown in Appendix B, $\Delta J_{\rm s}(x)$ is related to the excess quasiparticle current $\Delta J_{\rm q}$ as

$$\Delta J_{\rm s}(x) = -\Delta J_{\rm q}(x). \tag{38}$$

[Here, we set $\langle \alpha_j^\dagger \alpha_j \rangle = 1$ in Eq. (B31) assuming that one Bogoliubov excitation is injected.] As a result, the phase $\theta(x)$ is evaluated to be

$$\theta(x) = -m \int_{-\infty}^{x} dx' \frac{\Delta J_{\rm q}(x')}{n_{\rm s}(x')} \ . \tag{39}$$

Namely, the phase modulation is caused by the excess quasiparticle current $\Delta J_{\rm q}$. The assumption in Eq. (36) is valid as long as $\theta(x)$ is small, because the change in the amplitude of the condensate wave function gives higher-order corrections. Since $\theta(x)$ is inversely proportional to the number of condensate atoms N_0, as shown below, $\theta(x)$ is negligibly small, so that the assumption in Eq. (36) is justified. As discussed in Appendix B, the inclusion of the back-reaction effect of quasiparticles on condensates requires the modification of the GP equation as Eq. (B29). In the present case, the new condensate wave function including the back-reaction effect is perturbatively obtained with use of the ansatz in Eq. (36) without solving Eq. (B29).

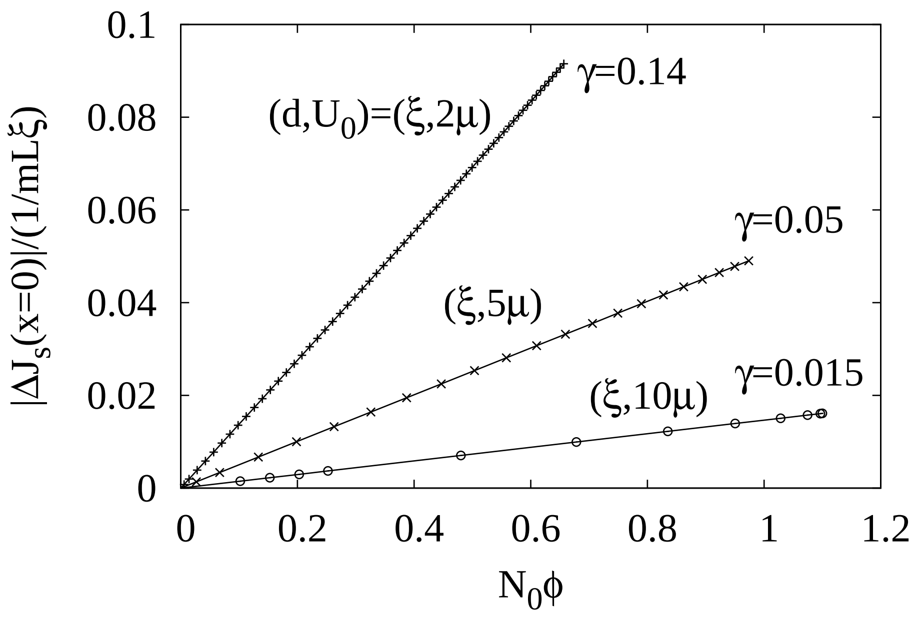

FIG. 6: Induced supercurrent $\Delta J_{\rm s}(x=0)$ as a function of the relative phase ϕ across the potential barrier. The slopes of the lines are $\gamma = 0.14$, 0.050, and 0.015 for $(d, U_0) = (\xi, 2\mu)$, $(\xi, 5\mu)$, and $(\xi, 10\mu)$, respectively.

Figure 4(b) shows $\theta(x)$ when the Bogoliubov phonon is injected from $x = -\infty$. The spatial variation in the phase $\theta(x)$ is remarkable near and in the barrier, where large excess current $\Delta J_{\rm q}$ is obtained [see Fig. 4(a)].

Figure 6 shows the magnitude of induced supercurrent at $x = 0$ as a function of $\phi \equiv \theta(-\infty) - \theta(x)$ [see Fig. 4(b)]. We clearly see that $|\Delta J_{\rm s}(x=0)|$ satisfies the ordinary Josephson current relation [31]

$$I(\phi) = I_J \sin\phi \simeq I_J \phi, \qquad (\phi \ll 1). \tag{40}$$

(In our case, since ϕ is proportional to the inverse of the total number of Bose-condensed particles N_0, so that $\phi \ll 1$.) The Josephson critical current I_J in the present case has the form,

$$I_J = \gamma \left(\frac{n_0}{m\xi} \right), \tag{41}$$

where γ is determined from the slope of the lines in Fig. 6. This result means that the Josephson critical current I_J may be evaluated from the analysis of quasiparticle tunneling without directly examining the Josephson current.

Finally, we remark that the tunneling properties of Bogoliubov excitation discussed in this section suggest an important role of Bogoliubov phonons on the fluctuation of the relative phase between two condensates at finite temperatures. When Bogoliubov phonons are excited on both sides of the barrier at finite temperatures, they tunnel through the potential barrier and twist the relative phase. This is expected to lead the fluctuation of the phase difference between the condensates on the left and right of the barrier. In particular, large phase fluctuations may be induced in the temperature region where the population of Bogoliubov phonon becomes dominant. This phase fluctuation due to the tunneling of Bogoliubov phonons could be observed in a BEC in a double-well potential, where the thermally induced fluctuations of the relative phase between two condensates were recently observed [32].

IV. TUNNELING BETWEEN CONDENSATES WITH DIFFERENT CONDENSATE DENSITIES

In Sec. III, we considered tunneling properties of Bogoliubov phonons through the rectangular potential barrier in the case when the left and right of the barrier have the same condensate densities. In this section, we consider the more general case when the condensate densities are different between the right and left of the barrier. This situation is achieved by simply imposing a uniform potential on the right side of the barrier, as

$$U(x) = U_0\theta\left(\frac{d}{2} - |x|\right) + U_1\theta\left(x - \frac{d}{2}\right). \tag{42}$$

In this case, the condensate density at $x \to \infty$ is given by

$$\tilde{n}_0 = \Psi_0(x=\infty)^2 = \begin{cases} \frac{1}{g}(\mu - U_1), & (0 \leq U_1 < \mu), \\ 0, & (U_1 \geq \mu). \end{cases} \tag{43}$$

In this section, we consider the case of $0 \leq U_1 < \mu$. The case of $U_1 \geq \mu$ will be discussed in Sec. V. The barrier potential, as well as the condensate wave function $\Psi_0(x)$, is schematically shown in Fig. 7.

To solve Bogoliubov equations (22) and (23), we construct the condensate wave function $\Psi_0(x)$, as well as the asymptotic solutions at $x = \pm\infty$. The former is analytically obtained

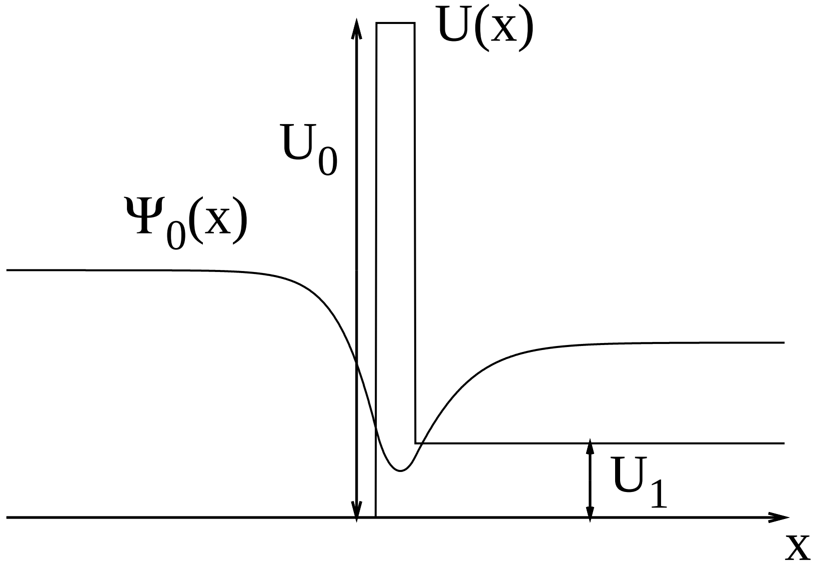

FIG. 7: Schematic of the system in the presence of the barrier potential in Eq. (42).

from Eq. (2) as

$$\bar{\Psi}_0(\bar{x}) = \begin{cases} \tanh\left[-\frac{1}{\sqrt{2}}\left(\bar{x}+\frac{\bar{d}}{2}\right)+\mathrm{arctanh}\gamma_L\right], & (x<-\frac{d}{2}), \\ \frac{\beta}{\mathrm{cn}\left(\sqrt{\frac{K^2+\beta^2}{2}}(\bar{x}-\bar{x}_0),q\right)}, & (|x|\le d/2) \\ \sqrt{1-\bar{U}_1}\tanh\left[\sqrt{\frac{1-\bar{U}_1}{2}}\left(\bar{x}-\frac{\bar{d}}{2}\right)+\mathrm{arctanh}\left(\frac{\gamma_R}{\sqrt{1-\bar{U}_1}}\right)\right], & (x>\frac{d}{2}), \end{cases} \tag{44}$$

where $\bar{U}_1 \equiv U_1/\mu$, $\gamma_L \equiv \bar{\Psi}_0(-\bar{d}/2)$, and $\gamma_R \equiv \bar{\Psi}_0(\bar{d}/2)$. x_0 satisfies the conditions $\bar{\Psi}(\bar{x}_0)=\beta$ and $d\Psi_0(x)/dx|_{x=x_0}=0$. x_0, β, γ_L, and γ_R are determined from the equations,

$$\gamma_R = \frac{\beta}{\mathrm{cn}\left(\sqrt{\frac{K^2+\beta^2}{2}}(\frac{\bar{d}}{2}-\bar{x}_0),q\right)}, \tag{45}$$

$$\gamma_L = \frac{\beta}{\mathrm{cn}\left(\sqrt{\frac{K^2+\beta^2}{2}}(\frac{\bar{d}}{2}+\bar{x}_0),q\right)}, \tag{46}$$

$$\gamma_R^2 = \frac{1}{2(\bar{U}_0-\bar{U}_1)}\left(\beta^4+2(\bar{U}_0-1)\beta^2+(1-\bar{U}_1)^2\right), \tag{47}$$

$$\gamma_L^2 = \frac{1}{2\bar{U}_0}\left(\beta^4+2(\bar{U}_0-1)\beta^2+1\right). \tag{48}$$

Equations (45)-(48) are derived from the boundary conditions at $x=\pm d/2$.

The asymptotic solutions of the Bogoliubov equations at $x = \pm\infty$ are obtained in the same manner as in Sec. III. Assuming that the Bogoliubov phonon with the energy $E = \sqrt{\varepsilon_p(\varepsilon_p + 2g\tilde{n})}$ is injected from $x = -\infty$, we have

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{ikx} + r \begin{pmatrix} a \\ b \end{pmatrix} e^{-ikx} + A \begin{pmatrix} -b \\ a \end{pmatrix} e^{\kappa x}, & (x \to -\infty), \\ \begin{pmatrix} u \\ v \end{pmatrix} = t \begin{pmatrix} a_R \\ b_R \end{pmatrix} e^{ik_R x} + B \begin{pmatrix} -b_R \\ a_R \end{pmatrix} e^{-\kappa_R x}, & (x \to \infty). \end{cases} \tag{49}$$

Here, k, κ, and (a, b) are given in Eqs. (29) and (30). The parameters appearing in the asymptotic solution at $x = \infty$ are given by

$$k_R = \sqrt{2m}\sqrt{\sqrt{E^2 + (g\tilde{n}_0)^2} - g\tilde{n}_0}, \tag{50}$$

$$\kappa_R = \sqrt{2m}\sqrt{\sqrt{E^2 + (g\tilde{n}_0)^2} + g\tilde{n}_0}, \tag{51}$$

$$\begin{pmatrix} a_R \\ b_R \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2L}\left(\frac{\sqrt{E^2+(g\tilde{n}_0)^2}}{E} + 1\right)} \\ \sqrt{\frac{1}{2L}\left(\frac{\sqrt{E^2+(g\tilde{n}_0)^2}}{E} - 1\right)} \end{pmatrix}. \tag{52}$$

Using the condensate wave function in Eq. (44), we numerically solve the Bogoliubov Eqs. (22) and (23) so as to satisfy Eq. (49). Once the wave function $(u(x), v(x))$, as well as r and t, are determined, we can calculate the transmission probability from the conserving energy flux $Q_{\rm q}$. The energy flux $Q_{\rm q}$ at $x \gg \xi$ is given by

$$Q_{\rm q} = \frac{k_R E}{mL}(a_R^2 + b_R^2)|t|^2. \tag{53}$$

The transmission (reflection) probability W (R) is conveniently defined as the ratio of the incident and transmitted (reflected) components of $Q_{\rm q}$. From Eqs. (34) and (53), we obtain

$$W = \frac{k_R(a_R^2 + b_R^2)}{k(a^2 + b^2)}|t|^2, \tag{54}$$

$$R = |r|^2. \tag{55}$$

Equations (54) and (55) satisfy the relation $R + W = 1$ because of the conservation of $Q_{\rm q}$ as proved in Appendix B.

We note that, when we calculate the transmission probability from the quasiparticle current, we obtain a different result from Eqs. (54) and (55). Using the expression for the

quasiparticle current at $x \gg \xi$,

$$J_{\rm q} = \frac{k_R}{mL}|t|^2 - \frac{2a_R b_R}{m}e^{-\kappa_R x}(\kappa_R {\rm Im}\left[tB^* e^{ik_R x}\right] + k_R {\rm Re}\left[tB^* e^{ik_R x}\right]), \tag{56}$$

and Eq. (33), we define the "transmission (reflection) probability" W_J (R_J) as the ratio of the incident and transmitted (reflected) components of $J_{\rm q}(x = \pm\infty)$. Then, we find

$$W_J = \frac{k_R}{k}|t|^2 = \frac{a^2+b^2}{a_R^2+b_R^2}W, \tag{57}$$

$$R_J = |r|^2 = R. \tag{58}$$

Since $W_J > W$, Eqs. (57) and (58) do *not* satisfy the condition $R_J + W_J = 1$, unless $U_1 = 0$. This is because of the fact that $J_{\rm q}$ is not conserved, as discussed in Sec. III and Appendix B. When $U_1 = 0$ (this case was discussed in Sec. III), the breakdown of the conservation of $J_{\rm q}$ is restricted to the region near the barrier. Namely, all the supplied component $\Delta J_{\rm q}$ is completely absorbed after the quasiparticle is transmitted in the right condensate, as shown in Fig. 4. As a result, the transmission probability, which is defined using $J_{\rm q}(x = \pm\infty)$, is not affected by this non-conserving character of $J_{\rm q}$. On the other hand, the fact of $W_J + R_J > 0$ when $U_1 > 0$ indicates that the non-conserving behavior of $J_{\rm q}$ remains even at $x = \infty$.

Figure 8 shows the calculated transmission probability W, W_J in Eq. (57), and the phase shift $\delta \equiv \arg(t)$. While the phase shift δ approaches 0 in the low-energy limit irrespective of the value of U_1, the perfect transmission ($W \to 1$ in the low-energy limit) is absent when $U_1 > 0$. In Ref. [19], Watabe and Kato have obtained the analytic expressions for W and δ in the low-energy limit for arbitrary potential barrier shape. According to their results, W and δ read [19]

$$W \to \frac{4\sqrt{1-\bar{U}_1}}{(1+\sqrt{1-\bar{U}_1})^2}, \qquad \delta \to 0, \qquad (E \to 0). \tag{59}$$

These results can be also obtained in the case of a δ-function potential barrier [33]. Our results in Fig. 8 are consistent with their earlier results in Eq. (59). Equation (59) shows that the transmission probability W becomes less than unity when $U_1 > 0$. As pointed out in Ref. [19], the low-energy behaviors of W and δ are determined only by the potential difference at $x = \pm\infty$ (U_1 in the present case), and they do not depend on the detail of the potential barrier in the middle.

In Fig. 8(b), one finds that W_J is remarkably enhanced to be larger than unity in the low-energy region. To see the relation of this large enhancement and the non-conserving

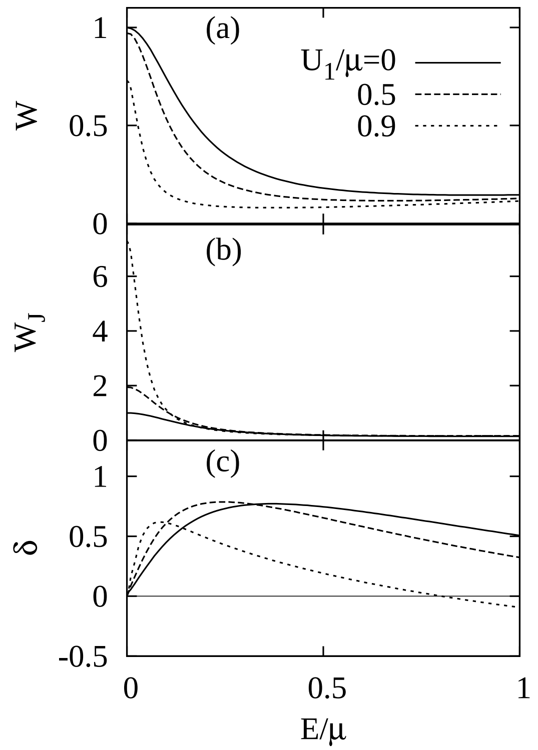

FIG. 8: Transmission probability W (a) calculated from the energy flux, and W_J (b) calculated from the quasiparticle current $J_{\rm q}$, and phase shift δ (c). We set $(d, U_0) = (\xi, 5\mu)$.

character of $J_{\rm q}$, we show the spatial variation in $J_{\rm q}$ in Fig. 9(a). Comparing this result with Fig. 4, we find that the excess quasiparticle current $\Delta J_{\rm q}$ remains finite even far away from the barrier ($x \gg d$) when $U_1 > 0$. This excess current is found to be supplied from the condensate through the source term $S(x)$ defined in Eq. (B14), as shown in Fig. 9(b). Figure 9(b) also shows that this supply dominantly occurs in front of the barrier ($-5 \lesssim x/\xi \lesssim 0$). (Note that the phonon is injected from $x = -\infty$.) As a result, W_J given by the ratio of incident and transmitted quasiparticle current is remarkably enhanced.

As discussed in Sec. III, the excess component $\Delta J_{\rm q}(x) = J_{\rm q}(x) - J_{\rm q}(x = -\infty)$ is cancelled out by the counter flow of supercurrent to conserve the total current. As shown in Fig. 9(c), the induced supercurrent remains finite even at $x \to \infty$, which is in contrast to the case of $U_1 = 0$, where $\Delta J_{\rm s}$ is only finite near the barrier. The reason for this can be considered as follows: as discussed in Sec. III, Bogoliubov phonons twist the condensate phase when they tunnel through a potential barrier. In addition, Bogoliubov phonons can be regarded as quantized oscillations of the phase of the condensate wave function [3]. Since the phase

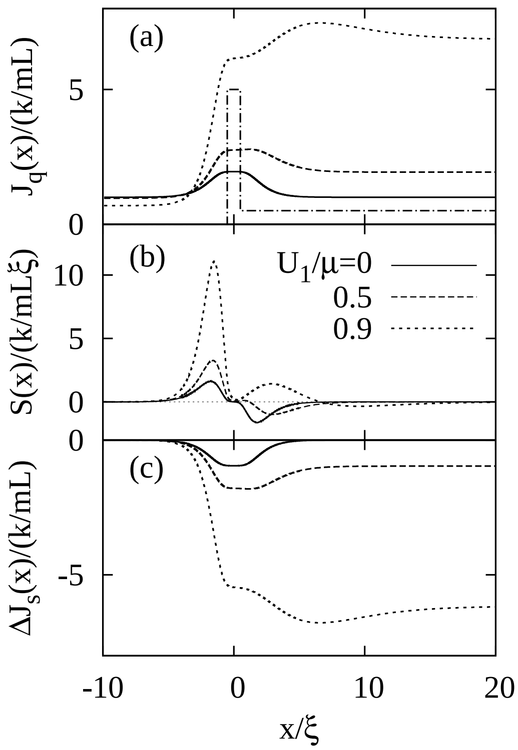

FIG. 9: Spatial variation in quasiparticle current $J_{\mathrm{q}}(x)$ (a), source term $S(x)$ (b), and induced supercurrent ΔJ_{s} (c) when $E = 0.01\mu$. We set $(d, U_0) = (\xi, 5\mu)$. The dash-dotted line in (a) shows the potential barrier $U(x)$ in units of μ when $U_1 = 0.5\mu$.

stiffness is weak on the right side of the barrier due to the small condensate density, the transmitted Bogoliubov phonons can easily twist the phase of the right condensate when $U_1 > 0$. This leads to the induction of counter superflow far away from the barrier. Indeed, $|\Delta J_{\mathrm{s}}(x \gg \xi)|$ and $J_{\mathrm{q}}(x \gg \xi)$ are larger for larger U_1, as shown in Fig. 9.

V. TUNNELING BETWEEN SUPERFLUID AND NORMAL REGIONS

In this section, we consider the case when the condensate density at $x \ll -\xi$ is absent. To realize this situation in a simple manner, we use the potential

$$U(x) = U_2\theta(-x) \tag{60}$$

with $U_2 \geq \mu$. In what follows, we call the negative x side the normal region and the positive x side the superfluid region. In the normal region, Bogoliubov excitations reduce to free

atoms, having the energy

$$E_p^{\mathrm{s}} = \varepsilon_p + (U_2 - \mu). \tag{61}$$

Here, we discuss two different tunneling problems, i.e., tunneling of atoms from the normal region to the superfluid region (N-S tunneling), and the tunneling of Bogoliubov excitations from the superfluid region to the normal region (S-N tunneling). These two cases enable us to study how free atoms are injected into a condensate and emitted from the surface of the condensate. We note that these tunneling problems are analogous to the quantum evaporation and condensation at a free surface in superfluid ^{4}He [34].

We first consider the N-S tunneling. The analytic solution of the GP Eq. (2) is given by

$$\bar{\Psi}_0(\bar{x}) = \begin{cases} -\frac{\lambda}{\sinh\left(\lambda\frac{\bar{x}}{\sqrt{2}} - C\right)}, & (x < 0), \\ \tanh\left(\frac{\bar{x}}{\sqrt{2}} + \mathrm{arctanh}\alpha\right), & (x \geq 0), \end{cases} \tag{62}$$

where $\alpha = 1/\sqrt{2\bar{U}_2}$, $\lambda = \sqrt{2(\bar{U}_2 - 1)}$, and $C = \log[(\lambda + \sqrt{\alpha^2 + \lambda^2})/\alpha]$.

The asymptotic solution of the Bogoliubov equations has the form

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{e^{ik_L x}}{\sqrt{L}} + r \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{e^{-ik_L x}}{\sqrt{L}} + A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{e^{\kappa_L x}}{\sqrt{L}}, & (x \to -\infty), \\ \begin{pmatrix} u \\ v \end{pmatrix} = t \begin{pmatrix} a \\ b \end{pmatrix} e^{ikx} + B \begin{pmatrix} -b \\ a \end{pmatrix} e^{-\kappa x}, & (x \to \infty), \end{cases} \tag{63}$$

where k, κ, and (a, b) are given in Eqs. (29) and (30). The wave numbers k_L and κ_L for $x \to -\infty$ are given by

$$k_L = \sqrt{2m}\sqrt{E - (U_2 - \mu)}, \tag{64}$$

$$\kappa_L = \sqrt{2m}\sqrt{E + (U_2 - \mu)}. \tag{65}$$

k_L and κ_L are propagating and localized waves for $x \to -\infty$, which are obtained by solving $E = \pm E_p^{\mathrm{s}}$ in terms of p. We note that the localized v component in Eq. (63) describes the (proximity) effect of the condensate in the normal region.

Using Eq. (63), we obtain the quasiparticle current J_{q}, as well as the energy flux of quasiparticles Q_{q}, in the normal region ($x \ll -\xi$) as

$$J_{\mathrm{q}} = \frac{k_L}{mL}(1 - |r|^2), \tag{66}$$

$$Q_{\mathrm{q}} = \frac{k_L E}{mL}(1 - |r|^2). \tag{67}$$

Since the localized v component in Eq. (63) does not give rise to any contribution to the currents, $J_{\rm q}$ and $Q_{\rm q}$ reduce to those of free atoms which satisfy the relation $Q_{\rm q} = EJ_{\rm q}$. From Eqs. (34) and (67), we obtain the transmission (reflection) probability W (R) as

$$W = L\frac{k}{k_L}(a^2+b^2)|t|^2, \tag{68}$$

$$R = |r|^2. \tag{69}$$

Since the energy flux is conserved as shown in Appendix B, they satisfy the condition $R + W = 1$. We also obtain the "transmission (reflection) probability" W_J (R_J) for quasiparticle current from Eqs. (33) and (66),

$$W_J = \frac{k}{k_L}|t|^2 = \frac{1}{L(a^2+b^2)}W, \tag{70}$$

$$R_J = |r|^2. \tag{71}$$

We again find that, W and W_J do not coincide with each other. Because of $W_J < W$, we obtain $R_J + W_J < 1$. This implies that the quasiparticle current decreases on the superfluid region. Since the ratio between W and W_J is given by $L(a^2+b^2) = \sqrt{E^2+(gn_0)^2}/E$, W and W_J become equal when $E/gn_0 \gg 1$.

Figure 10 shows the transmission probability W obtained from the energy flux and the phase shift $\delta \equiv \arg(t)$. We also show the quasiparticle transmission probability $W_J^{\rm NS}$ in Fig. 11 (a). In Figs. 10 and 11, we note that the origin of E is taken to be $U_2 - \mu$, because atoms are perfectly reflected when $E < U_2 - \mu$, leading to vanishing $W_J^{\rm NS}$ and W. We find that both W and W_J decrease with decreasing E, and W and W_J approach 0 when $E \to U_2 - \mu$, while δ approaches a positive value in the limit of $E \to U_2 - \mu$. Thus, the anomalous tunneling behavior does not occur in the present case.

Figure 12(a) shows the spatial variation in quasiparticle current $J_{\rm q}(x)$, source term $S(x)$ [defined by Eq. (B14)], as well as the induced supercurrent $\Delta J_{\rm s}(x)$. The existence of transmitted component of $J_{\rm q}(x)$ shows that the incident current of free atoms from the normal region is converted into the Bogoliubov excitations inside the condensate. Furthermore, one finds that $J_{\rm q}(x)$ decreases near the surface at $x \simeq 0$, and the supercurrent $\Delta J_{\rm s}(x)$ is induced around the surface. The source term $S(x)$ becomes negative near the surface of the superfluid region reflecting the behaviors of $J_{\rm q}(x)$ and $\Delta J_{\rm s}(x)$. These phenomena indicate that injected atoms are Bose condensed in the superfluid region, which give rise to the supercurrent $\Delta J_{\rm s}$. The condition $R_J + W_J < 1$ reflects the fact that a part of the incident current of free atoms

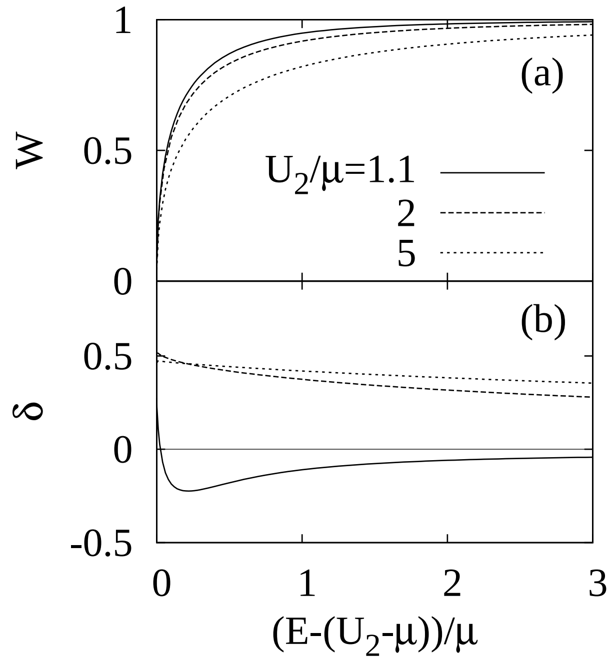

FIG. 10: Transmission probability W obtained from the energy flux (a) and phase shift δ (b) in both the N-S and S-N tunneling cases.

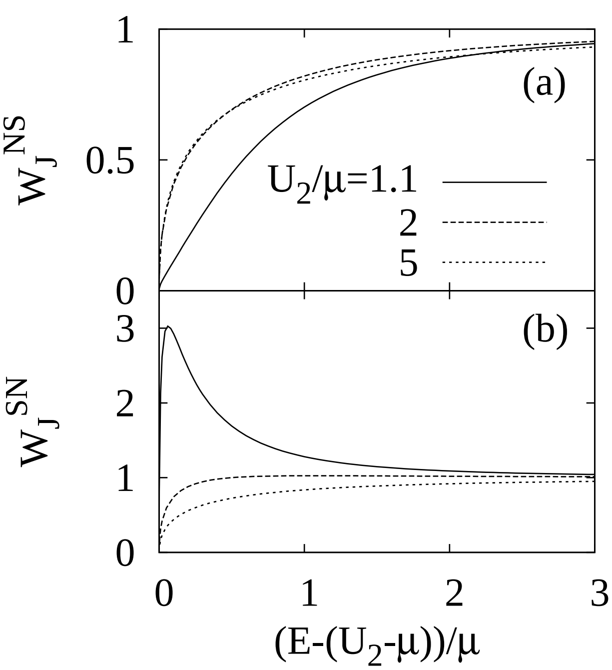

FIG. 11: Transmission probabilities obtained from the quasiparticle current in the N-S (W_J^{NS}) (a) and S-N (W_J^{SN}) (b) tunneling cases.

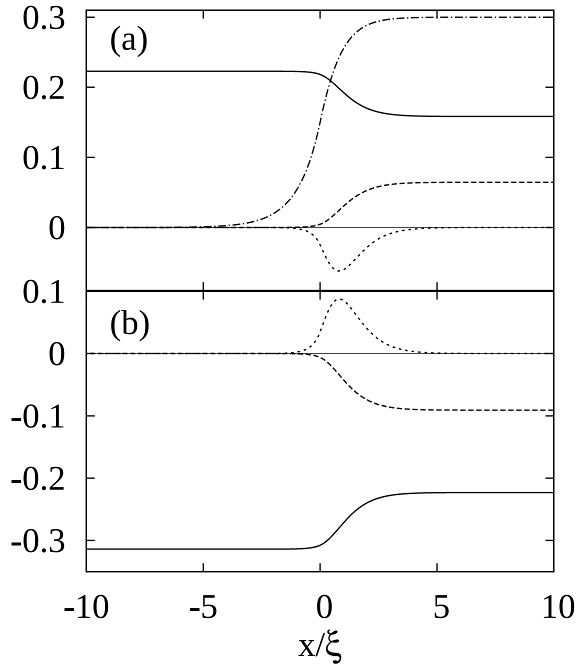

FIG. 12: Spatial variation in quasiparticle current $J_{\rm q}(x)$ (solid line), induced supercurrent $\Delta J_{\rm s}(x)$ (dashed line), and source term $S(x)$ (dotted line), when $U_2 = 2\mu$ and $E = 1.01\mu$ for the N-S tunneling (a) and S-N tunneling (b). Current density and source term are in units of k_L/mL and $k_L/mL\xi$ in (a), and k/mL and $k/mL\xi$ in (b), respectively. The dash-dotted line in (a) indicates the condensate wave function $\Psi_0(x)$ in units of $\sqrt{n_0}/0.3$.

is converted to supercurrent inside the condensate. The supercurrent $\Delta J_{\rm s}$ decreases as E increases, because the character of produced Bogoliubov phonon becomes close to that of single-particle excitation, as E increases. As a result, W_J approaches W when $E \gg gn_0$.

We next consider the S-N tunneling. Assuming that the incident Bogoliubov mode comes from $x = +\infty$, the asymptotic solutions (u, v) for $x \to \pm\infty$ are given by

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{e^{-ik_L x}}{\sqrt{L}} + B \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{e^{\kappa_L x}}{\sqrt{L}}, & (x \to -\infty), \\ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-ikx} + r \begin{pmatrix} a \\ b \end{pmatrix} e^{ikx} + A \begin{pmatrix} -b \\ a \end{pmatrix} e^{-\kappa x}, & (x \to \infty). \end{cases} \tag{72}$$

Using Eq. (72), the quasiparticle current $J_{\rm q}$ and energy flux of quasiparticles $Q_{\rm q}$ in the

normal region ($x \ll -\xi$) are calculated as

$$J_{\rm q} = \frac{-k_L}{mL}|t|^2, \tag{73}$$

$$Q_{\rm q} = \frac{-k_L E}{mL}|t|^2. \tag{74}$$

From Eqs. (34) and (74), we obtain the transmission (reflection) probability W (R) obtained from the energy flux as

$$W = \frac{1}{L(a^2+b^2)}\frac{k_L}{k}|t|^2, \tag{75}$$

$$R = |r|^2. \tag{76}$$

Equations (75) and (76) again satisfy the condition $R + W = 1$. From Eqs. (33) and (73), we obtain the transmission (reflection) probability W_J (R_J) for quasiparticle current as

$$W_J = \frac{k_L}{k}|t|^2 = L(a^2+b^2)W, \tag{77}$$

$$R_J = |r|^2. \tag{78}$$

In contrast to the N-S tunneling, it is clear from Eq. (77) that $W_J > W$, which leads to the condition $R_J + W_J > 1$. This implies that the quasiparticle current is supplied around the surface at $x \simeq 0$. When $E \gg gn_0$, W_J reduces to W.

It can be generally shown for the Bogoliubov coupled Eqs. (3) and (4) that W and δ are both independent of whether the incident wave comes from $x = -\infty$ (N-S tunneling) or $x = +\infty$ (S-N tunneling) [33]. Hence, W and δ in the S-N tunneling case are the same as those in the case of N-S tunneling in Fig. 10.

The transmission probability for quasiparticle current $W_J^{\rm SN}$ is shown in Fig. 11(b). We find that $W_J^{\rm SN}$ is enhanced to be greater than unity at low energies, due to the factor $L(a^2+b^2) = \sqrt{E^2+(gn_0)^2}/E$ in Eq. (77).

When the Bogoliubov phonons propagate toward the S-N phase boundary, Fig. 12(b) shows that atoms evaporate from the surface. We also find that $J_{\rm q}(x)$ changes near the surface and the supercurrent $\Delta J_{\rm s}$ is induced, which flows toward the boundary. $S(x)$ becomes positive around the surface of the superfluid region, reflecting the behavior of $J_{\rm q}(x)$ and $\Delta J_{\rm s}(x)$. This supercurrent $\Delta J_{\rm s}(x)$ is considered to be induced by the reflected Bogoliubov mode, which twists the condensate phase in the superfluid region. The fact of $R_J + W_J > 1$ reflects that $J_{\rm q}(x)$ increases during the tunneling through the S-N phase boundary.

VI. CONCLUSIONS

To summarize, we have investigated tunneling effects of Bogoliubov excitations at $T = 0$. We have extended our previous work to the case when the condensate densities are different on the left and right of the barrier. Within the frame work of the Bogoliubov theory, we have evaluated the transmission probability, phase shift as well as the energy flux and quasiparticle current carried by Bogoliubov excitations. We showed that, while the energy flux is conserved, the quasiparticle current is not conserved. The excess quasiparticle current is actually cancelled out by the counterflow of supercurrent, which is induced by the back-reaction effects of Bogoliubov phonons on the condensate. In the case of a rectangular potential barrier, we directly showed that the induced supercurrent satisfies the Josephson relation with respect to the twisted phase by Bogoliubov phonons. When the condensate has different densities on the left and right of the barrier, the supercurrent is induced in the region far from the barrier potential. We also studied the tunneling of atoms from the normal region to the superfluid region, as well as the tunneling of excitations from the superfluid region to the normal region. In the former case, we showed that supercurrent is induced inside a condensate by injecting free atoms from outside. In the latter case, we found that atoms evaporate from the superfluid-normal state phase boundary, when Bogoliubov excitations propagate toward the surface of the superfluid region. We think these results can be of interest for the investigation of Bogoliubov mode and its connection to the superfluidity of BECs in ultracold atomic gases.

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APPENDIX A: FORMALISM OF WEAKLY INTERACTING BOSE GASES

In this appendix, we summarize the formalism of weakly interacting Bose gases developed in [1, 24, 25, 26, 35]. We introduce approximations for inhomogeneous Bose condensates including the Bogoliubov approximation used in this paper.

We consider an interacting Bose gas described by the Hamiltonian,

$$\hat{K} = \int d\boldsymbol{r}\ \hat{\psi}^\dagger(\boldsymbol{r}) \left(-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \mu \right) \hat{\psi}(\boldsymbol{r}) + \frac{g}{2} \int d\boldsymbol{r}\ \hat{\psi}^\dagger(\boldsymbol{r}) \hat{\psi}^\dagger(\boldsymbol{r}) \hat{\psi}(\boldsymbol{r}) \hat{\psi}(\boldsymbol{r}), \tag{A1}$$

where $\hat{\psi}(\boldsymbol{r})$ is the Bose field operator, μ is the chemical potential, and $U(\boldsymbol{r})$ is an external potential. We assume a contact interaction between atoms $g\delta(\boldsymbol{r} - \boldsymbol{r}')$ with the coupling constant $g = 4\pi a_s/m$, where $a_s > 0$ is the s-wave scattering length.

In the Bose condensed phase, we divide the field operator $\hat{\psi}(\boldsymbol{r})$ into the sum of the condensate wave function $\Psi_0(\boldsymbol{r}) = \langle \hat{\psi}(\boldsymbol{r}) \rangle$ and the fluctuation part $\delta\hat{\psi}$ as

$$\hat{\psi}(\boldsymbol{r}) = \Psi_0(\boldsymbol{r}) + \delta\hat{\psi}(\boldsymbol{r}). \tag{A2}$$

Substituting Eq. (A2) into Eq. (A1), we approximately evaluate the cubic and quartic terms with respect to $\delta\hat{\psi}$ and $\delta\hat{\psi}^\dagger$ as

$$\delta\hat{\psi}^\dagger \delta\hat{\psi} \delta\hat{\psi} \ \simeq\ 2\tilde{n}\delta\hat{\psi} + \tilde{m}\delta\hat{\psi}^\dagger, \tag{A3}$$

$$\delta\hat{\psi}^\dagger \delta\hat{\psi}^\dagger \delta\hat{\psi} \delta\hat{\psi} \ \simeq\ 4\tilde{n}\delta\hat{\psi}^\dagger \delta\hat{\psi} + \tilde{m}^* \delta\hat{\psi}\delta\hat{\psi} + \tilde{m}\delta\hat{\psi}^\dagger \delta\hat{\psi}^\dagger, \tag{A4}$$

where $\tilde{n}(\boldsymbol{r}) = \langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \rangle$ is the non-condensate density, and $\tilde{m}(\boldsymbol{r}) = \langle \delta\hat{\psi} \delta\hat{\psi} \rangle$ is the so-called anomalous average [35]. In this mean-field approximation, Eq. (A1) reduces to

$$\hat{K} \ =\ \hat{K}_0 + \hat{K}_1 + \hat{K}_2, \tag{A5}$$

$$\hat{K}_0 \ =\ \int d\boldsymbol{r}\ \Psi_0^* \hat{T} \Psi_0 + \frac{g}{2} \int d\boldsymbol{r}\ |\Psi_0|^2, \tag{A6}$$

$$\hat{K}_1 \ =\ \int d\boldsymbol{r} \left[\left(\hat{T}\Psi_0 + g(|\Psi_0|^2 + 2\tilde{n})\Psi_0 + g\tilde{m}\Psi_0^* \right) \delta\hat{\psi}^\dagger + \text{h.c.} \right], \tag{A7}$$

$$\hat{K}_2 \ =\ \int d\boldsymbol{r} \left[\delta\hat{\psi}^\dagger \left(\hat{T} + 2g(|\Psi_0|^2 + \tilde{n}) \right) \delta\hat{\psi} + \frac{g}{2} \left((\Psi_0^2 + \tilde{m}) \delta\hat{\psi}^\dagger \delta\hat{\psi}^\dagger + \text{h.c.} \right) \right]. \tag{A8}$$

Here, $\hat{T} \equiv -\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \mu$. From the condition that the linear term in terms of $\delta\psi$ and $\delta\psi^\dagger$ vanishes, we obtain the (generalized) Gross-Pitaevskii equation [24, 26],

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + g(|\Psi_0|^2 + 2\tilde{n}) \right] \Psi_0 + g\tilde{m}\Psi_0^* = \mu\Psi_0\ . \tag{A9}$$

The quadratic term $\hat{K}_2$ in Eq. (A8) can be diagonalized by the Bogoliubov transformation [25]

$$\delta\hat{\psi}(\boldsymbol{r}) = \sum_j \left[u_j(\boldsymbol{r})\hat{\alpha}_j - v_j(\boldsymbol{r})^* \hat{\alpha}_j^\dagger \right], \tag{A10}$$

$$\delta\hat{\psi}^\dagger(\boldsymbol{r}) = \sum_j \left[u_j(\boldsymbol{r})^*\hat{\alpha}_j^\dagger - v_j(\boldsymbol{r}) \hat{\alpha}_j \right], \tag{A11}$$

where $\hat{\alpha}_j^\dagger$ is the creation operator of a Bogoliubov excitation in the jth state, which obeys the bosonic commutation relations,

$$[\hat{\alpha}_i, \hat{\alpha}_j^\dagger] = \delta_{i,j},\ \ [\hat{\alpha}_i, \hat{\alpha}_j] = [\hat{\alpha}_i^\dagger, \hat{\alpha}_j^\dagger] = 0. \tag{A12}$$

Diagonalization of $\hat{K}_2$ is achieved when $(u_j(\boldsymbol{r}), v_j(\boldsymbol{r}))$ satisfy the following generalized Bogoliubov equations [24, 25, 35]:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + 2g(|\Psi_0|^2 + \tilde{n}) - \mu \right] u_j - g(\Psi_0^2 + \tilde{m}) v_j = E_j u_j, \tag{A13}$$

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + 2g(|\Psi_0|^2 + \tilde{n}) - \mu \right] v_j - g((\Psi_0^*)^2 + \tilde{m}^*) u_j = -E_j v_j. \tag{A14}$$

Then, we have

$$\hat{K} = \hat{K}_0 + \sum_j E_j \hat{\alpha}_j^\dagger \hat{\alpha}_j - \sum_j E_j \int d\boldsymbol{r}\ |v_j|^2 . \tag{A15}$$

The last term in Eq. (A15) is the so-called quantum depletion, describing the non-condensate due to the repulsive interaction between atoms. It remains finite even at $T = 0$, where $\langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle = 0$.

We note that Eq. (A9) involves terms originating from excitations ($2g\tilde{n}\Psi_0$ and $g\tilde{m}\Psi_0^*$). This reflects the fact that the condensate and excitations affect each other. In the Bogoliubov approximation, both $\tilde{n}$ and $\tilde{m}$ are neglected, so that effects of Bogoliubov excitations on the condensate are not taken into account. Equation (A9) without $\tilde{n}$ and $\tilde{m}$ is the ordinary static GP equation [24, 26], while Eqs. (A13) and (A14) without $\tilde{m}$ and $\tilde{n}$ are the Bogoliubov coupled equations. The Bogoliubov approximation is valid at very low temperatures where $\tilde{n}$ and $\tilde{m}$ are very small.

The approximation keeping both $\tilde{n}$ and $\tilde{m}$ is the Hartree-Fock-Bogoliubov approximation [35]. This approximation is valid at finite temperatures where non-condensate fluctuation cannot be neglected. In this approximation, however, the excitation spectrum in a uniform system has an energy gap [35]. This is inconsistent with the Hugenholtz-Pines theorem [36],

which states that the excitation spectrum must be gapless in the BEC phase. Keeping $\tilde{n}$ but neglecting $\tilde{m}$ is referred to as the Popov approximation [3, 35]. This approximation is also considered to be valid at finite temperatures. Since it yields a gapless excitation spectrum, it has been widely used in the study of BEC at finite temperatures [3, 35, 37]. Tunneling properties of Bogoliubov excitations at finite temperatures have been also studied using the Popov approximation [13].

APPENDIX B: CONSERVATION LAWS FOR BOGOLIUBOV EXCITATIONS

In this appendix, we discuss the conservation laws in terms of quasiparticle current and energy flux associated with Bogoliubov excitations. In the Bogoliubov mean-field approximation, the total number density $n \equiv \langle \hat{\psi}^\dagger \hat{\psi} \rangle$ and total current density $\boldsymbol{J} \equiv (1/m)\mathrm{Im}\langle \hat{\psi}^\dagger \nabla \hat{\psi} \rangle$ are, respectively, given by

$$n = n_{\mathrm{s}} + \sum_j \left(n_{u_j} + n_{v_j}\right) \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle + \sum_j n_{v_j}, \tag{B1}$$

$$\boldsymbol{J} = \boldsymbol{J}_{\mathrm{s}} + \sum_j \left(\boldsymbol{J}_{u_j} - \boldsymbol{J}_{v_j}\right) \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle - \sum_j \boldsymbol{J}_{v_j}. \tag{B2}$$

In this appendix, the index for eigenstates j is explicitly written. Note that in our tunneling problem of Bogoliubov excitation, each eigenstate in Eqs. (B1) and (B2) corresponds to Bogoliubov excitation with energy E injected from $x = -\infty$ or $x = \infty$. Here, $n_{\mathrm{s}} \equiv |\Psi_0|^2$ describes the condensate density and

$$\boldsymbol{J}_{\mathrm{s}} = \frac{1}{m}\mathrm{Im}(\Psi_0^* \nabla \Psi_0) \tag{B3}$$

is the supercurrent density carried by the condensate. n_{u_j}, n_{v_j}, $\boldsymbol{J}_{u_j}$, and $\boldsymbol{J}_{v_j}$ are, respectively, given by

$$n_{u_j} = |u_j|^2, \tag{B4}$$

$$n_{v_j} = |v_j|^2, \tag{B5}$$

$$\boldsymbol{J}_{u_j} = \frac{1}{m}\mathrm{Im}(u_j^* \nabla u_j), \tag{B6}$$

$$\boldsymbol{J}_{v_j} = \frac{1}{m}\mathrm{Im}(v_j^* \nabla v_j). \tag{B7}$$

The total number density n and total current density $\boldsymbol{J}$ satisfy the continuity equation

$$\partial_t n + \nabla \cdot \boldsymbol{J} = 0. \tag{B8}$$

Since the second terms in Eqs. (B1) and (B2) describe the quasiparticle contributions, the quasiparticle density $n_{\mathrm{q},j}$ and quasiparticle current $\boldsymbol{J}_{\mathrm{q},j}$ are, respectively, given by

$$n_{\mathrm{q},j} = n_{u_j} + n_{v_j}, \tag{B9}$$

$$\boldsymbol{J}_{\mathrm{q},j} = \boldsymbol{J}_{u_j} - \boldsymbol{J}_{v_j}. \tag{B10}$$

Equations (B9) and (B10) show that both the quasiparticle density $n_{\mathrm{q},j}$ and current $\boldsymbol{J}_{\mathrm{q},j}$ consist of two components originating from u_j and v_j. We note that the current density of $v$component appears as $-\boldsymbol{J}_{v_j}$ in Eq. (B10). Thus, in a uniform system, a Bogoliubov phonon is accompanied by two current components, $\boldsymbol{J}_{u_j} = (\boldsymbol{p}/m)a^2$ and $-\boldsymbol{J}_{v_j} = -(\boldsymbol{p}/m)b^2$, where a and b are given in Eq. (30). Hence, the v component flows in the oppose direction to the u component. Indeed, these counterpropagating currents were recently observed [38]. The last terms in Eqs. (B1) and (B2) describe effects of quantum depletion.

To derive the continuity equation for quasiparticles, it is convenient to use the time-dependent Bogoliubov equations [39] for $(u(\boldsymbol{r},t), v(\boldsymbol{r},t))$,

$$i\tau_3\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{h} & -g\Psi_0^2 \\ -g(\Psi_0^*)^2 & \hat{h} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{B11}$$

where $\hat{h} \equiv -\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + 2g|\Psi_0|^2 - \mu$. Equation (B11) reduces to Eqs. (3) and (4) in the stationary state, $(u(\boldsymbol{r},t), v(\boldsymbol{r},t)) = e^{-iE_jt}(u_j(\boldsymbol{r}), v_j(\boldsymbol{r}))$.

Using Eq. (B11), one obtains the continuity equations for u_j and v_j, as

$$\partial_t n_{u_j} + \nabla\cdot\boldsymbol{J}_{u_j} = \frac{S_j}{2}, \tag{B12}$$

$$\partial_t n_{v_j} - \nabla\cdot\boldsymbol{J}_{v_j} = \frac{S_j}{2}, \tag{B13}$$

where

$$S_j = -4g\mathrm{Im}\left(\Psi_0^2 u_j^* v_j\right). \tag{B14}$$

Thus, the continuity equation for quasiparticles is given by

$$\partial_t n_{\mathrm{q},j} + \nabla\cdot\boldsymbol{J}_{\mathrm{q},j} = S_j\ . \tag{B15}$$

In Eq. (B15), S_j works as a source term. This means that the total number of quasiparticles is *not conserved* when $S_j \neq 0$. In a uniform system, one finds $S_j = 0$, so that the number of quasiparticles is conserved. On the other hand, since the source term S_j is finite near the potential barrier in our tunneling problem, the number of quasiparticles is *not* conserved.

We next consider the energy flux. For this purpose, we define the energy density operator $\hat{\rho}$ as

$$\hat{\rho} = \hat{\psi}^\dagger \hat{T} \hat{\psi} + \frac{g}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi}, \tag{B16}$$

where $\hat{T}$ is defined below Eq. (A8). Using the Heisenberg equation $i\partial_t \hat{\psi} = \hat{T}\hat{\psi} + g\hat{\psi}^\dagger \hat{\psi}\hat{\psi}$, one obtains the continuity equation for energy density $\hat{\rho}$ as,

$$\partial_t \hat{\rho} + \nabla \cdot \hat{\boldsymbol{Q}} = 0. \tag{B17}$$

Here, $\hat{\boldsymbol{Q}}$ is the energy flux operator, defined by

$$\begin{aligned} \hat{\boldsymbol{Q}} &= \frac{i}{2m} \left[(\nabla \hat{\psi}^\dagger) \left(\hat{T}\hat{\psi} + g\hat{\psi}^\dagger \hat{\psi}\hat{\psi} \right) - \text{h.c.} \right] \\ &= -\frac{1}{m} \text{Re} \left[(\nabla \hat{\psi}^\dagger)(\partial_t \hat{\psi}) \right]. \end{aligned} \tag{B18}$$

Substituting Eq. (A2) into $\rho = \langle \hat{\rho} \rangle$ and retaining terms up to $O(\delta\hat{\psi}^2)$, we obtain

$$\begin{aligned} \rho &= \Psi_0^* \hat{T} \Psi_0 + \frac{g}{2} |\Psi_0|^4 + \langle \delta\hat{\psi} \hat{T} \delta\hat{\psi} \rangle \\ &+ \frac{g}{2} \left(\Psi_0^2 \langle \delta\hat{\psi}^\dagger \delta\hat{\psi}^\dagger \rangle + 4|\Psi_0|^2 \langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \rangle + (\Psi_0^*)^2 \langle \delta\hat{\psi} \delta\hat{\psi} \rangle \right). \end{aligned} \tag{B19}$$

In obtaining Eq. (B19), we have used $\langle \delta\hat{\psi} \rangle = 0$. Using Eqs. (A10), (A13), and (A14), we obtain

$$\rho = \rho_0 + \sum_j E_j \left(n_{u_j} - n_{v_j} \right) \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle - \sum_j E_j n_{v_j} - \frac{i}{4} \sum_j S_j, \tag{B20}$$

where

$$\rho_0 = \Psi_0^* \hat{T} \Psi_0 + \frac{g}{2} |\Psi_0|^4 \tag{B21}$$

is the condensate energy density. Since the energy density ρ is a real quantity, the last term in Eq. (B20) must vanish, which gives

$$\sum_j S_j = 0. \tag{B22}$$

The energy flux $\boldsymbol{Q} \equiv \langle \hat{\boldsymbol{Q}} \rangle$ can be also calculated in the same manner. The result is

$$\boldsymbol{Q} = \boldsymbol{Q}_0 + \sum_j E_j (\boldsymbol{J}_{u_j} + \boldsymbol{J}_{v_j}) \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle + \sum_j E_j \boldsymbol{J}_{v_j}. \tag{B23}$$

Here,

$$\boldsymbol{Q}_0 = \frac{i}{2m} \left[\left(\hat{T}\Psi_0 + g(|\Psi_0|^2 + 2\tilde{n})\Psi_0 + g\tilde{m}\Psi_0^* \right) (\nabla \Psi_0^*) - \text{c.c.} \right] \tag{B24}$$

is interpreted as the energy flux carried by the condensate. Actually, $\boldsymbol{Q}_0$ identically vanishes when Ψ_0 satisfies the (generalized) GP equation.

The second terms in Eqs. (B20) and (B23) describe the quasiparticle contributions. Thus, the energy density for quasiparticles $\rho_{\mathrm{q},j}$ and energy flux for quasiparticles $\boldsymbol{Q}_{\mathrm{q},j}$ are, respectively, given by

$$\rho_{\mathrm{q},j} = E_j(n_{u_j} - n_{v_j}), \tag{B25}$$

$$\boldsymbol{Q}_{\mathrm{q},j} = E_j(\boldsymbol{J}_{u_j} + \boldsymbol{J}_{v_j}). \tag{B26}$$

Equation (B25) shows that the v component has a negative energy density $-E_j n_{v_j}$. In Eq. (B26), the v component appears as $+E_j\boldsymbol{J}_{v_j}$, which is in contrast to $\boldsymbol{J}_{\mathrm{q},j}$ in Eq. (B10), where the v component appears as $-\boldsymbol{J}_{v_j}$. This is because the v component has a negative energy $-E_j$ and counterpropagating current density $-\boldsymbol{J}_{v_j}$. In contrast to the nonconserved quasiparticle number density in Eq. (B15), the continuity equation with respect to the energy density has no source term, as

$$\partial_t \rho_{\mathrm{q},j} + \nabla\cdot\boldsymbol{Q}_{\mathrm{q},j} = 0. \tag{B27}$$

Namely, $\boldsymbol{Q}_{\mathrm{q},j}$ is conserved everywhere in the stationary state.

To examine the origin of the source term S_j in Eq. (B15), it is convenient to consider the divergence of Eq. (B2) in the stationary state,

$$\nabla\cdot\boldsymbol{J} = \nabla\cdot\boldsymbol{J}_{\mathrm{s}} + \sum_j S_j\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle. \tag{B28}$$

In obtaining Eq. (B28), we have used Eqs. (B12)-(B14) and (B22). The static GP Eq. (2) guarantees the conservation of the supercurrent ($\nabla\cdot\boldsymbol{J}_{\mathrm{s}} = 0$), so that Eq. (B28) contradicts with the conservation of the total current $\boldsymbol{J}$ obtained from Eq. (B8), unless the last term in Eq. (B28) vanishes identically.

This inconsistency arises because effects of quasiparticles on condensates (back-reaction effect) are completely neglected in the Bogoliubov approximation. This problem can be solved by including quasiparticle contribution to the condensate in the GP equation as

$$\left(-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) + g|\Psi_0|^2\right)\Psi_0 - 2g\sum_j u_j v_j^*\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle\Psi_0^* = \mu\Psi_0\ . \tag{B29}$$

In this modified GP equation, the last term on the left-hand side originates from the anomalous average $\tilde{m}$ in Eq. (A9) as

$$g\langle\delta\hat{\psi}\delta\hat{\psi}\rangle\Psi_0^* = -2g\sum_j u_j v_j^*\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle\Psi_0^* - g\sum_j u_j v_j^*\Psi_0^*. \tag{B30}$$

Using Eq. (B29), the conservation of the supercurrent ($\nabla \cdot \boldsymbol{J}_{\rm s} = 0$) is modified to be

$$\nabla \cdot \boldsymbol{J}_{\rm s} = -\sum_j S_j \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle. \tag{B31}$$

Substituting Eq. (B31) into Eq. (B28), we obtain the expected conservation of the total current $\nabla \cdot \boldsymbol{J} = 0$. In the case of the one-dimensional model we are using in this paper, when we integrate Eq. (B31) in terms of x from $-\infty$ to x, we obtain Eq. (38).

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